Canonical transformation and Hamilton – Jacobi theory
In several problems, we may need to change one set of position and momentum coordinates into another set of position and momentum coordinates. Suppose that \( q \) and \( p \) are the old position and momentum coordinates and \( Q_k \) and \( P_k \) are the new ones.

Let these coordinates be related by the following transformations:

\[
P_k = P_k(p_1, \ldots, p_n, q_1, \ldots, q_n, t)
\]

\[
Q_k = Q_k(p_1, \ldots, p_n, q_1, \ldots, q_n, t)
\]

Now if there exist a Hamiltonian \( H' \) in the new coordinates such that

\[
P_k^\cdot = -\frac{\partial H'}{\partial Q_k} \quad \text{and} \quad Q_k^\cdot = \frac{\partial H'}{\partial P_k}
\]

Where, \( H' = P_k Q_k^\cdot - L' \)

and \( L' \) substituted in the Hamilton's principle
\[ \delta \int L' \, dt = 0 \]

Gives the correct equations of motion in terms of the new coordinate $P_k$ and $Q_k$, then these transformation are known as canonical transformation.
**CONDITION FOR CANONICAL TRANSFORMATION.**

- Suppose \( F = F(q_k, Q_k) \) then obviously \( \frac{\partial F}{\partial t} = 0 \) and \( H = H' \)

\[
p_k = \frac{\partial F}{\partial q_k} \quad \text{and} \quad P_k = -\frac{\partial F}{\partial Q_k}
\]

Also

\[
dF = \sum \frac{\partial F}{\partial q_k} dq_k - \sum \frac{\partial F}{\partial Q} dQ_k
\]

\[
dF = \sum p_k dq_k - \sum P_k dQ_k
\]

The L.H.S of above equation is exact differential, hence for given transformation to be canonical, the R.H.S. i.e. \( \sum p_k dq_k - \sum P_k dQ_k \) must be an exact differential.
Those transformations in which the new set of coordinates $(Q, P)$ differ from old set $(q, p)$ by infinitesimals i.e. $Q = q + \delta q$ and $P = p + \delta p$ are called **infinitesimal contact transformations**.

\[ F_2 = \sum q P + \varepsilon G(q, P) \]

\[ p = \frac{\partial F_2}{\partial q_k} = P_k + \varepsilon \frac{\partial G}{\partial q_k}, \]

\[ Q = \frac{\partial F_2}{\partial P_k} = q_k + \varepsilon \frac{\partial G}{\partial P_k}, \]

\[ H' = H \]

\[ Q_k - q_k = \delta q_k = \varepsilon \frac{\partial G}{\partial P_k}, \quad P_k - p_k = \delta p_k = -\varepsilon \frac{\partial G}{\partial q_k} \]

\[ \delta q_k = \varepsilon \frac{\partial G}{\partial P_k} \quad \delta p_k = -\varepsilon \frac{\partial G}{\partial q_k} \]
In special case $dt = \varepsilon$, $G = H$,

$$\delta q_k = dt \frac{\partial H}{\partial P_k} = dt q \cdot_k = dq_k$$

$$\delta p_k = -dt \frac{\partial H}{\partial q_k} = dt p \cdot_k = dp_k$$

The motion of the system in a small time ‘$dt$’ can be described by an infinitesimal canonical transformation generated by the Hamiltonian $H$ of the system.
POISSON’S BRACKETS

- If the functions $F$ and $G$ depend upon the position coordinate $q$, momentum coordinate $p$ and time $t$, Poisson bracket of $F$ and $G$ defined as

$[F, G]_{q,p} = \sum \left( \frac{\partial F}{\partial q_k} \frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k} \frac{\partial G}{\partial q_k} \right)$

- For brevity we may drop subscripts $q, p$ and write only $[F, G]$

- The total time derivative of function $F$ can be written as

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum \left[ \left( \frac{\partial F}{\partial q_k} \right) q^* k + \frac{\partial F}{\partial p_k} \right] P^* k$$

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum \left[ \left( \frac{\partial F}{\partial q_k} \right) \frac{\partial H}{\partial p_k} + \frac{\partial F}{\partial p_k} \right] \frac{\partial H}{\partial q_k}$$
In Poisson's bracket form,
\[
\frac{dF}{dt} = \frac{\partial F}{\partial t} + [F, H]
\]

if \( \frac{dF}{dt} = 0 \), or \( \frac{\partial F}{\partial t} + [F, H] = 0 \)

now if \( F \) does not depend on time explicitly, \( \frac{\partial F}{\partial t} = 0 \) and then condition for \( F \) to be constant of motion is obtained to be

\[
[F, H] = 0
\]

In other words Poisson bracket with Hamiltonian vanishes is constant of motion
Thank You